Mathematical Preliminaries

Announcement

- I am looking for a grader for this class.
- Tools for viewing PostScript files.

Last Time

- Sets / Sequences
- Functions / Relations
- Graphs

Strings

- Strings of characters
- Alphabet : any finite set, Σ and Γ

$$\Sigma = \{0, 1, 2, 3, x, y\}$$

- A string over an alphabet: a finite sequence of symbols from the alphabet

$$w = x1211y3$$

- Length of a string w: $|w|$
- Empty string: ϵ

Strings

- Reverse of w: w^R
- Substring
- Concatenation

$$w = w_1 w_2 w_3 \quad v = v_1 v_2$$

$$\longrightarrow wv = w_1 w_2 w_3 v_1 v_2$$

- Lexicographic ordering of strings

$$\Sigma = \{0, 1\}, \quad (\epsilon, 0, 1, 00, 01, 10, 11, 000, \cdots)$$

Languages

- Language : a set of strings

$$\{00, 010, 111, 10001\}$$

- Finite / infinite
- Kleene closure of alphabet A: A^*, the set of all strings of all lengths over A

$$\{0\}^* = \{\epsilon, 0, 00, 000, 0000, \cdots\}$$

$$\{00 \cup 1\}^* = \{\epsilon, 1, 00, 001, 100, 0000, \cdots\}$$

- A language L over the alphabet A is a subset of A^*

$$L \subseteq A^*$$

Proofs

- Proof: a convincing logical argument that a statement is true.
  - convincing in an absolute sense
- Methods of proof
  - The pigeonhole principle
  - Proof by construction
  - Proof by contradiction
  - Proof by induction

The Pigeonhole Principle

- If there are n pigeonholes, $n+1$ or more pigeons, and every pigeon occupies a hole,

  then some hole must have at least two pigeons.

- Example
  - To show a 9-color pie chart on a 8-color monitor, two parts must have the same color.

Proof by Construction

- Prove a particular type of objects exists by constructing the object.
- Example
  - For each even number $n > 2$, there exists a 3-regular graph with n nodes.
  - Proof: construct a graph $G = (V, E)$ with n nodes as follows.

    $V = \{0, 1, \cdots, n-1\}$

    $E = \{\{i, i+1\} \mid \text{for } 0 \leq i \leq n-2\}$

    $\cup \{\{n-1, 0\}\}$

    $\cup \{\{i, i+n/2\} \mid \text{for } 0 \leq i \leq n/2-1\}$

Proof by Contradiction

- Assume a theorem is false and then show that this assumption leads to a false consequence.
- Example
  - The language $L = \{00 \cup 1\}^*$ contains strings with an even number of 1's.
  - Proof: Assume L contains only strings with an odd number of 1's. However, L contains the string 11, which contradicts with the assumption.

Proof by Induction

- A proof by induction has a predicate P,
  - a basis — $\exists k, P(k)$ is true
  - an induction hypothesis — for some $n \geq k$, $P(k), P(k+1), \cdots, P(n)$ are true.
  - an inductive step — $P(n+1)$ is true given the induction hypothesis.
- Example
  - $\forall n \geq 0, S(n) = \Sigma_{j=1}^{n} j = n(n+1)/2$.
  - Proof:
    * Basis: $S(0) = 0$.
    * Induction hypothesis: $S(k) = k(k+1)/2$ for $k = 0, 1, \cdots, n$.
    * Inductive step: $S(n+1) = S(n) + n + 1$